# Effiziente Algorithmen

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# Assignment 11

Take home: 06/25/2012
Submit: 07/02/2012

*Note*: It is understood that all of your statements have to be proven correct.

*Note*: Solutions may be submitted by email. Solutions submitted after the lecture will not be graded.

*Note*: The maximum score for this assignment is 16. All exceeding points will be added to the total score of all assignments.

## Exercise 11.1. (8) — *A Top-10 algorithm*

We want to sample from a finite state space $S$. We construct a Markov Chain $C$ on $S$ such that the stationary distribution of a random walk on $S$ coincides with a given distribution $\pi$, where $\pi_s > 0$ for all $s \in S$. (For instance we may choose $S$ to be the set of independent sets of a given graph and $\pi$ the uniform distribution on these independent sets).

To construct $C$ we define a neighborhood $N(s) \subseteq S$ for all $s \in S$ such that the resulting graph is connected. To complete the construction for $C$ define the transition matrix by

$$P(s,t) = \begin{cases} \frac{1}{d} \cdot \frac{1}{2} \cdot \min(1, \frac{\pi(t)}{\pi(s)}) & \text{if } t \in N(s), \\ \text{the remaining probability} & \text{if } t = s. \end{cases}$$

where $d = \max_{s \in S} |N(s)|$.

Prove that $\pi$ is indeed $C$'s unique stationary distribution.

## Exercise 11.2. (16) — *Mixing and Coupling*

We investigate how fast a random walk on an ergodic Markov Chain $C$ with state space $S$ approximates its stationary distribution. Therefore we measure the distance of a $k$-step walk to $C$'s stationary distribution $\pi$ by

$$||P^k, \pi|| = \max_{s \in S} \frac{1}{2} \sum_{t \in S} |P^k(s,t) - \pi(t)|.$$

Utilizing this distance we now define the *$\epsilon$-mixing time* of $C$ to be

$$m(\epsilon) = \min\{k : \forall k' \geq k : ||P^{k'}, \pi|| \leq \epsilon\}.$$

A Markov Chain is said to be *rapidly mixing* if $m(\epsilon) = O(\text{polylog}(|S|) \cdot \ln(1/\epsilon))$.

We now discuss the method of *coupling* to bound mixing times. A coupling $\mathfrak{C}$ for the Markov chain $C$ is a random walk on the state space $S \times S$. In particular, for all starting states $(s,t)$ of $\mathfrak{C}$, the transition probabilities must satisfy the following conditions:

- If $(X_k, Y_k)_{k=0}^{\infty}$ is the sequence of random variables for states of the coupling $\mathfrak{C}$, then $(X_k)_{k=0}^{\infty}$ and $(Y_k)_{k=0}^{\infty}$ are sequences of random variables for the states of walks defined by the original chain $C$ starting in $s$ and $t$, respectively.
- Whenever $X_k$ and $Y_k$ coincide, so do $X_{k+1}$ and $Y_{k+1}$.

Define the *coupling time* $T_{s,t} = \min\{k \mid X_k = Y_k \text{ if } \mathfrak{C} \text{ starts in } (s,t)\}$. You may assume that

$$m(\epsilon) \leq e \cdot \ln(1/\epsilon) \cdot \max_{s,t \in S} E(T_{s,t}),$$

holds, where $e$ is Euler's number.

We want to bound the mixing time for the following Markov chain defining random walks on the $n$-dimensional hypercube $H_n$. In the current state $x_1 \ldots x_n$ choose a random position $i \in \{1, \ldots, n\}$ and a random bit $b$ and jump to state $x_1 \ldots x_{i-1} b x_{i+1} \ldots x_n$.

Define a coupling for the chain such that $\max_{s,t \in S} E(T_{s,t})$ is small. In particular show that the chain is rapidly mixing.

*Hint*: Observe that whenever the two walks reach the same state at the same time $k$, then $X_{k+1} = Y_{k+1}$.