Graph-theoretic Problems

Parallel algorithms for fundamental graph-theoretic problems:

- We already used a parallelization of dynamic programming to solve the all-pairs-shortest-path problem.
- Here we are interested in:
  - determining connected components,
  - computing minimal spanning trees,
  - finding maximal independent sets and
  - coloring graphs.
- Symmetry breaking and the use of randomness turn out to be important new techniques.

Connected Components

Let $G = (V, E)$ be an undirected graph.

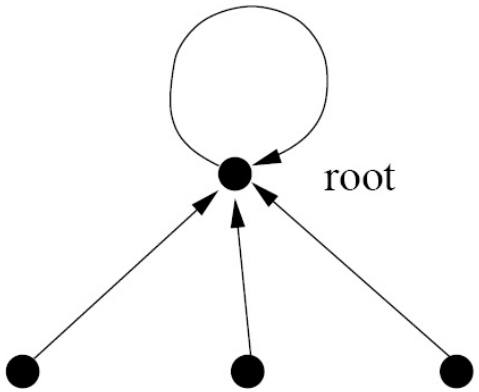
- A subset $W \subseteq V$ is connected iff any two nodes in W are connected by a path in G.
- A connected subset $W \subseteq V$ is a connected component iff no super set of W is connected.

Determine all connected components of G.

- Any two connected components are node-disjoint.
- Sequentially, connected components are computed in linear time $O(|V| + |E|)$:
  - The set of all nodes reachable from a node v is a connected component.
  - Use depth-first search to determine the set of reachable nodes.
- The problem: there are no efficiently parallelizable graph traversals.

Stars

- Throughout we represent the nodes of a component by a star:
  - exactly one root is distinguished,
  - all nodes including the root point to the root.



- Initially we have to assume that each node is its own connected component.
  - We start with n singleton loops, one singleton loop per node.
- Why stars? Merge two stars by
  - making one root point to the other root and
  - applying pointer jumping to restore the star property.

Symmetry Breaking

We may merge two stars S_1 and S_2, if there is an edge from a node in S_1 to an node in S_2.

- There will be many edges connecting a given star to other stars. Which merger should we perform?
  - If we merge S_1 and S_2, as well as S_2 and S_3 as well as . . . , then our approach is sequential.
  - We should forbid long chains, but merging, say, S_1 with $S_2, S_3, \ldots$ is OK, since we can easily represent the result by a single star.
- To decide on mergers immediately, we "break symmetry".
  - Each star chooses a gender at random.
  - Each male star chooses one female star and attaches himself below the female star.
  - How to choose the female star?

The Algorithm

(0) We have one processor per edge and one processor per node.

(1) for $i = 1$ to n pardo parent$[i] = i$. // We create singleton loops.

(2) while (at least one edge of G connects different stars) do

  (a) each root w chooses a gender at random and assigns its gender to its children.

  (b) if a processor e is responsible for an edge connecting a male star m with a female star f, then e tries to write f into the register of m.

  (c) The **arbitrary-write resolution** chooses one female star $f(m)$ for each male star m. The root of m is made to point to the root of $f(m)$.

  (d) for $i = 1$ to n pardo
  parent$[i] =$ parent[parent$[i]$];
  // One round of pointer jumping repairs the star property.

The Analysis I

- Correctness:
  - Stars are merged only if they belong to the same connected component.
  - Say that an edge is alive iff it connects two different stars.
    (If no edge is alive, then all connected components are found.)
- We utilize concurrent read (when determining whether an edge connects a male and a female star) as well as concurrent write (when edge processors try to write into the register of a male star).
- Running time:
  - The time per iteration is constant.
  - How many iterations do we have?

The Analysis II

Say that a star is alive
iff there is an edge connecting it to a **different** star.

- The root r of a fixed alive star S disappears with probability at least $\frac{1}{4}$. Why?
  - Let $\{u, v\}$ be an edge connecting S with a different star T.
  - With probability $\frac{1}{4}$: S becomes male and T female.
    If this happens, then r will disappear.
- Fix some node w and determine the probability p_w that w does not disappear as root after $5 \cdot \log_2 n$ iterations. Then

$$\begin{aligned} p_w &\leq \left(1 - \frac{1}{4}\right)^{5 \cdot \log_2 n} = \left(\frac{3}{4}\right)^{5 \cdot \log_2 n} = \left(\frac{243}{1024}\right)^{\log_2 n} \\ &\leq \left(\frac{1}{4}\right)^{\log_2 n} = 2^{-2 \cdot \log_2 n} = n^{-2}. \end{aligned}$$

Summary

- A fixed node survives $5 \cdot \log_2 n$ iterations with probability at most n^{-2}.
- Some node survives $5 \cdot \log_2 n$ iterations with probability at most $n \cdot n^{-2} = n^{-1}$.

The algorithm runs in expected time $O(\log_2 n)$ with $n + m$ processors on a CRCW-PRAM in the arbitrary mode.
In particular $5 \cdot \log_2 n$ iterations of the while-loop suffice with probability at least $1 - \frac{1}{n}$.

- Why expected time $O(\log_2 n)$?