ON GRAPHS THAT DO NOT CONTAIN A THOMSEN GRAPH

W.G. Brown

(received February 7, 1966)

1. A Thomsen graph [2, p.22] consists of six vertices partitioned into two classes of three each, with every vertex in one class connected to every vertex in the other; it is the graph of the "gas, water, and electricity" problem [1, p.206]. (All graphs considered in this paper will be undirected, having neither loops nor multiple edges.)

We define $g(n)$ to be the largest integer m for which there exists a graph of n vertices and $m-1$ edges containing no Thomsen graph; (it may, however, contain a subdivision of a Thomsen graph). It has been shown by Kővári, Sós, and Turán [7] that

$$g(n) < \frac{3n + 2^{1/3} n^{5/3}}{2} . \tag{1.1}$$

This has been improved by Znám [8]; but his result still yields the same result in the limit, viz.

$$\limsup_{n \to \infty} n^{-5/3} g(n) \leq 2^{-2/3} .$$

Kővári et al. [7] and Erdős [5] have conjectured that

$$g(n) > cn^{5/3} \tag{1.2}$$

for some positive constant c. In this paper we prove that conjecture correct.

The author wishes to acknowledge several enlightening conversations with his colleagues, Dr. R. Westwick and Dr. W. McWorter, during the course of this research.

Canad. Math. Bull. vol. 9, no. 3, 1966.

2. A lower bound for $g(n)$.

Let p be an odd prime. We construct a graph G whose vertices are the p^3 points of the affine geometry $EG(3, p)$, i.e. ordered triples $x = (x_1, x_2, x_3)$ of elements of $GF(p)$.

Define $S(x)$ to be the set of points y of $EG(3, p)$ for which

$$\sum_{i=1}^{3} (x_i - y_i)^2 = \alpha \tag{2.1}$$

where α is a fixed element of $GF(p)$ chosen to be a non-zero quadratic residue if $p \equiv 3 \pmod 4$, and a quadratic non-residue otherwise. Then, by a well known theorem of Lebesgue [3, p. 325] the number of points in $S(x)$ is $p^2 - p$. We shall connect vertices x and y of G by an edge if and only if

$$y \in S(x) \text{ ; or, what is equivalent, } x \in S(y) .$$

This graph has p^3 vertices, each of valency $p^2 - p$; thus $(p^5 - p^4)/2$ edges.

Suppose that G contains a Thomsen graph with vertices $a, a', a''; b, b', b''$ and edges connecting each a with each b. The points b, b', b'' must lie in $S(a) \cap S(a') \cap S(a'')$. Thus $w = b, b'$, or b'' are three solutions of the equations

$$\Sigma (a_i - w_i)^2 = \Sigma (a_i' - w_i)^2 = \Sigma (a_i'' - w_i)^2 = \alpha$$

hence also of the equations of the radical planes of these spheres, viz.

$$2A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \Sigma \begin{pmatrix} a_i^2 - a_i'^2 \\ a_i'^2 - a_i''^2 \\ a_i''^2 - a_i^2 \end{pmatrix} \tag{2.2}$$

where

$$A = \begin{pmatrix} a_1 - a_1' & a_2 - a_2' & a_3 - a_3' \\ a_1' - a_1'' & a_2' - a_2'' & a_3' - a_3'' \\ a_1'' - a_1 & a_2'' - a_2 & a_3'' - a_3 \end{pmatrix}$$

A is evidently singular. As the a's are distinct, the rank of A is 1 or 2. Thus, if there exists a Thomsen graph as described, either the a's or the b's are collinear. That this is impossible is a consequence of the following lemma.

(2.3) LEMMA. No three points of $S(x)$ are collinear.

Proof. By a suitable translation we can arrange that the line of points pass through the origin. Suppose

$$y = \tau a \quad (a \neq (0,0,0);\ \tau \text{ ranges over } GF(p)) \tag{2.4}$$

meets $S(x)$ in more than two points. Substituting (2.4) in (2.1) yields the quadratic equation in τ

$$(\Sigma\, a_i^2)\tau^2 - 2\,(\Sigma\, a_i x_i)\tau + \Sigma\, x_i^2 = \alpha$$

which can have more than two solutions only if

$$\Sigma\, a_i^2 = 0 \tag{2.5}$$

$$\Sigma\, a_i x_i = 0 \tag{2.6}$$

$$\Sigma\, x_i^2 = \alpha \tag{2.7}$$

Since $a \neq (0,0,0)$ we can assume without limiting generality that $a_1 \neq 0$. Then

$$a_1^2\alpha = a_1^2\,\Sigma\, x_i^2 = (-a_2x_2 - a_3x_3)^2 + a_1^2(x_2^2 + x_3^2) \quad \text{by (2.6)}$$

$$= -(a_3x_2 - a_2x_3)^2 \quad \text{by (2.5).}$$

This contradicts the choice of α since -1 is a quadratic residue if $p \equiv 1 \pmod 4$ and a quadratic non-residue otherwise.

We have thus shown that, for odd primes p, (2.8) $$g(p^3) > \frac{p^5 - p^4}{2} .$$

For any ε in the interval $0 < \varepsilon < 1$ there is an integer N_ε such that for all $n > N_\varepsilon$ there exists a prime p for which $n^{1/3} > p > (1 - \varepsilon)^{1/5} n^{1/3}$ [7, p. 371]. Hence, since g is non-decreasing,

$$g(n) \geq g(p^3) > \frac{p^5 - p^4}{2} > \frac{n^{5/3}}{2} - \frac{\varepsilon n^{5/3} + n^{4/3}}{2}$$

for all $n > N_\varepsilon$, from which (1.2) follows immediately. Moreover, $\liminf_{n \to \infty} n^{-5/3} g(n) \geq 1/2$. We cannot prove the existence of $\lim_{n \to \infty} n^{-5/3} g(n)$.

3. Graphs without quadrangles.

Define $f(n)$ to be the maximum integer m for which there exists a graph G with n vertices and m edges containing no quadrilateral. It is proved in [7] that $\limsup_{n \to \infty} f(n) n^{-3/2} = 1/2$. Using the following construction it can be shown that $\lim_{n \to \infty} f(n) n^{-3/2} = 1/2$. (The existence of this limit, with a different value, was conjectured by Erdős in [5] and elsewhere. This construction has also been found independently by Rényi, Mrs. Turán, and Erdős, and will appear in a forthcoming paper.)

Construct for each odd prime q a graph G as follows: The vertices of G are the points of $PG(2, q)$, i.e. the lines through the origin in $EG(3, q)$. Two vertices $(x_1, x_2, x_3) = (\tau a_1, \tau a_2, \tau a_3)$, $(x_1, x_2, x_3) = (\sigma a_1, \sigma b_2, \sigma b_3)$ are connected by an edge in G if and only if a and b are distince points of $EG(3, q)$ not collinear with the origin, and

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = 0.$$

Then q^2 vertices have valency $q + 1$, and $q + 1$ vertices have valency q. Thus G has $q^2 + q + 1$ vertices and

$\frac{(q^2+q+1)^{3/2}}{2} + 0(q^2)$ edges, but no quadrilateral. We leave the proof of the latter to the reader.

REFERENCES

1. C. Berge, Théorie des graphes. (Paris, 1958).

2. H.S.M. Coxeter and W.O.J. Moser, Generators and Relations for Discrete Groups, (Springer Verlag, 1957).

3. L.E. Dixkson, Theory of Numbers, Volume II, (Chelsea Publishing Company, 1952).

4. P. Erdős, Extremal problems in graph theory, in Theory of Graphs and its Applications, edited by M. Fiedler (Prague-New York-London, 1964).

5. P. Erdős, On extremal problems of graphs and generalized graphs, Israel J. Math. 2, (1964), 183-190.

6. G.H. Hardy and E.M. Wright, An introduction to the theory of numbers (Fourth Edition), (Oxford, 1960).

7. T. Kővári, V.T. Sós, P. Turán, On a problem of K. Zarankiewicz. Coll. Math. 3, (1955), 50-57.

8. Š. Znám, Two improvements of a result concerning a problem of K. Zarankiewicz, Colloq. Math. 13, (1965), 255-258.

University of British Columbia